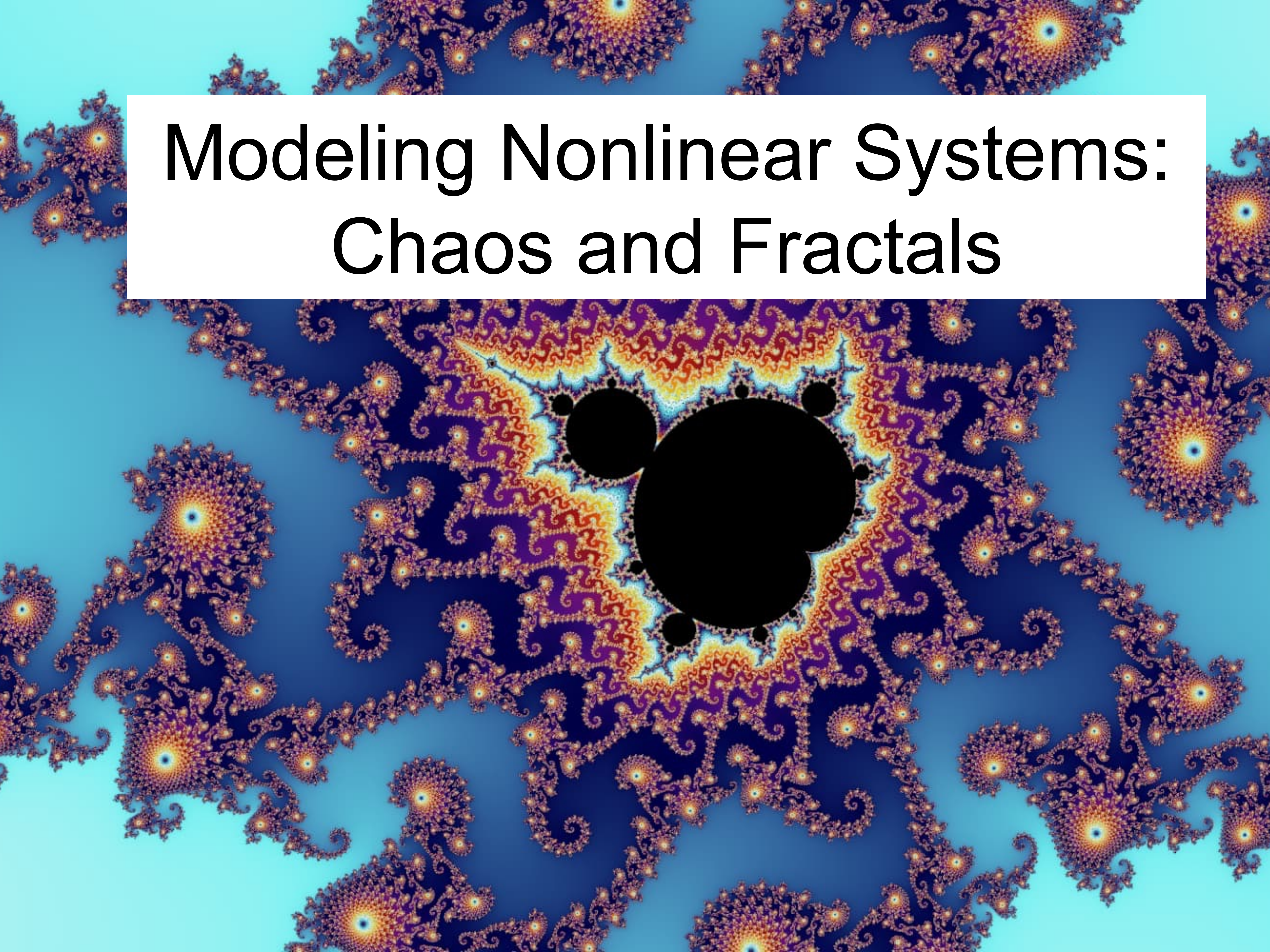
Modeling Nonlinear Systems: Chaos and Fractals



Nonlinear Dynamics: time evolution equations (e.g., Newton's 2nd law)/field equations contain nonlinear terms

There are many nonlinear systems in physics:

Mechanical systems: driven/coupled oscillators

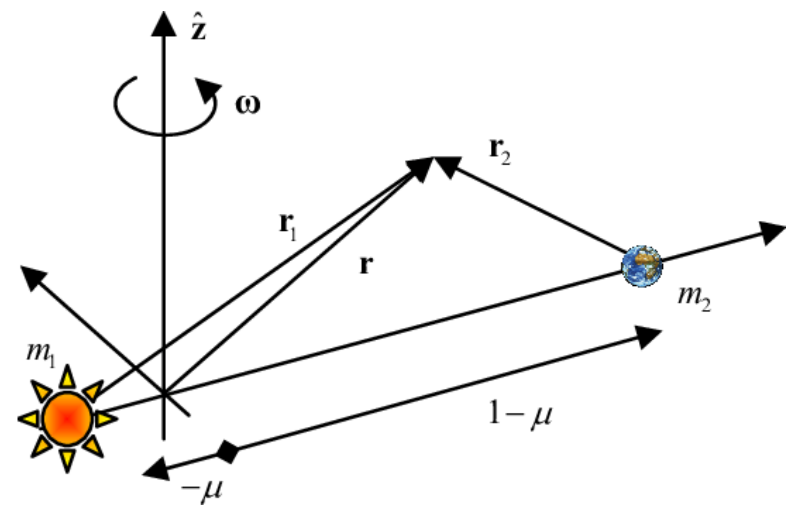
Astrophysics: **three-body problem**; stability of solar system; KAM theorem

General Relativity: Einstein equations

Electronics: nonlinear circuits

Hydrodynamics: turbulences

Climate science: weather patterns



Some nonlinear systems outside of physics :

Biology: predator-prey models

Chemistry: certain reactions…

Physiology: heart arrhythmias

Economics: stock market trends

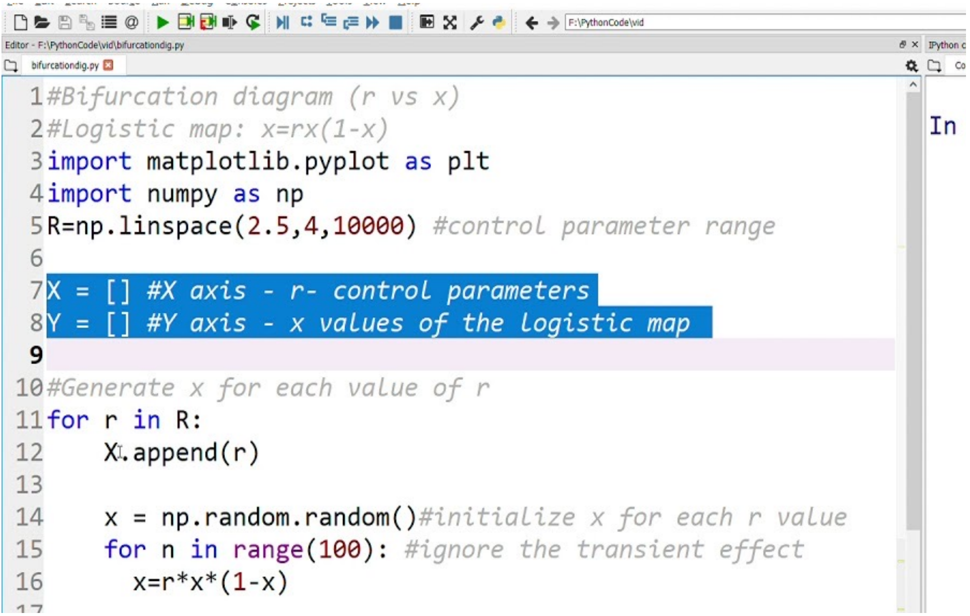
Simple example: the **logistic equation** for population growth, modeled as a differential equation…

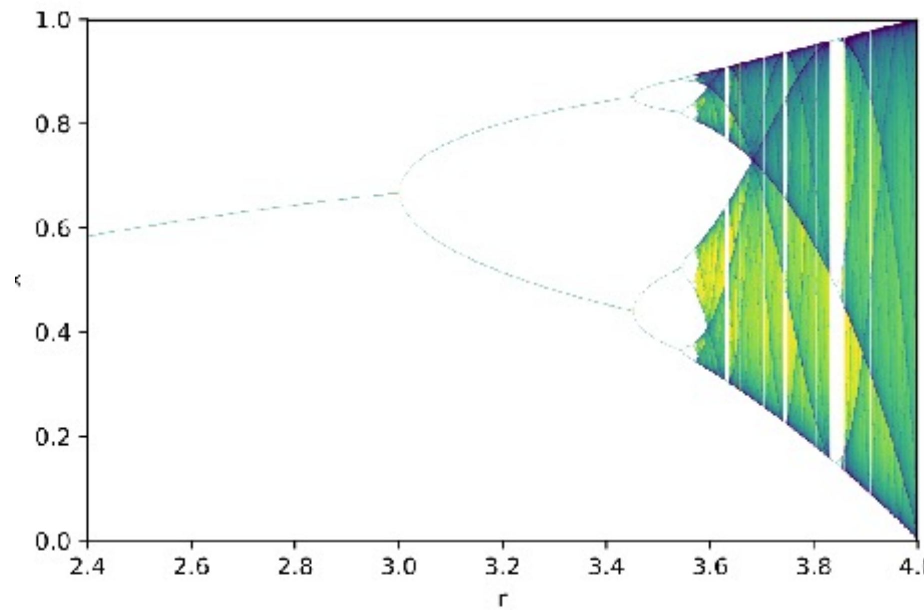
$$\frac{dx}{dt} = r\,x(1-x)$$

…or as a difference equation…

$$x_{n+1} = r\,x_n(1-x_n)$$

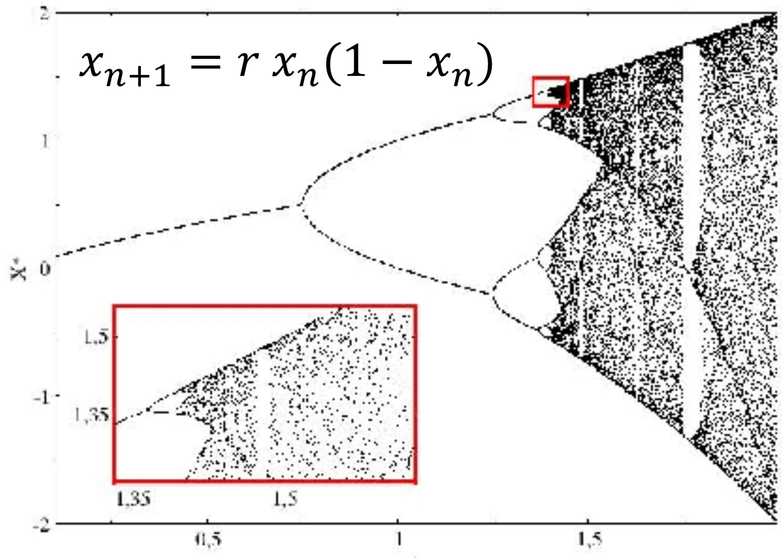
…graphed in Python:





Bifurcations for $r > 3$, chaotic long-term behavior for $r \geq 3.57$ ..., Initial value problems (IVPs) usually have a unique solution; they are *deterministic*. Characteristic for nonlinear dynamical systems is their *Sensitive Dependence on Initial Conditions* (SDIC): similar initial conditions produce very different long-term behaviors. This is known as **deterministic chaos**.

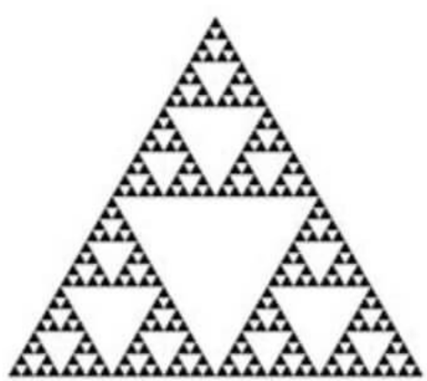
The geometry of deterministic chaos often shows self-similar patterns: **fractals**.



Fractals are everywhere:

In nature:

In geometry:



In algebra: